A VARIATIONAL EXPRESSION FOR THE SCATTERING MATRIX OF A STEP DISCONTINUITY IN A COAXIAL LINE AND ITS APPLICATION TO THE STUDY OF A MULTIMODE COAXIAL TEM CELL*

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ABSTRACT

A variational expression, for the scattering matrix of a step discontinuity in a multimode coaxial transmission line, is obtained and the result is used to analyze the transmission characteristics of a coaxial TEM cell.

1. Introduction

A step discontinuity, in a coaxial transmission line, of the type shown in Fig. 1a, may be analyzed by replacing it with an equivalent shunt susceptance, provided the transverse dimensions of the coaxial line are small enough that at the operating frequency only the TEM mode can propagate. However as the frequency is increased higher order modes will start propagating, in one or both sections of the line, and any analysis must consider the excitation and the intercoupling of these modes due to the prsence of the discontinuity. A convenient method of analysis is to develop a scattering matrix representation of a junction, with (M + N + 2) ports, M and N being the number of propagating higher order modes in the left and the right sections of the line. In this paper, a variational formulation of the scattering matrix elements is obtained. Numerical results for the transmission of TEM mode to and from a coaxial TEM cell capable of propagating TEM as well as TM_{01} mode are then given.

2. Formulation

Consider the geometry shown in Fig. 1a, and the scattering matrix representation (Fig. 1b) of the junction, where the dimensions of the line are assumed to be such that only the TEM mode can propagate in the left hand side section ($z \leq 0$) and two modes, namely the TEM and the $TM_{0,1}$, can propagate in the right hand section ($z \geq 0$). Let the incident and the reflected magnetic fields $H^{+}_{\phi i}$ and $H^{-}_{\phi i}$ associated with the propagating modes, with a time dependence of $\exp(i\omega t)$, be given by

$$H^{(\pm)}_{\phi 1} = \begin{pmatrix} +a_1 \\ -b_1 \end{pmatrix} (2\pi\zeta_0)^{-1/2} Z_0(r_1, r_3; \rho) e^{\pm i\beta_{0,A} z} \qquad (2.1a)$$

$$H^{(\pm)}_{\phi 2} = \begin{pmatrix} -a_2 \\ +b_2 \end{pmatrix} (2\pi\zeta_0)^{-1/2} Z_0(r_2, r_4; \rho) e^{\pm i\beta_{0,B} z} \qquad (2.1b)$$

$$H^{(\pm)}_{\phi 3} = \begin{pmatrix} -a_3 \\ +b_3 \end{pmatrix} (2\pi\zeta_0\beta_{1,B}/\beta_{0,B})^{-1/2} Z_1(r_2, r_4; \rho) \cdot e^{\pm i\beta_{1,B} z} \qquad (2.1c)$$

where $Z_p(a,b;\rho)$ represents the orthonormal function for the magnetic field of a TM_{0p} mode, with $p = 0$ referring to the TEM mode, in a coaxial line, with inner and outer conductors of radii a and b respectively; $\zeta_0 = \sqrt{\mu/\varepsilon}$ is the free space characteristic impedance; $\beta_{p,A}$ and $\beta_{p,B}$ are the propagation constants, of the TM_{0p} mode in regions A ($z \leq 0$) and B ($z \geq 0$) respectively, given by

$$\beta_{p,\substack{A\\B}} = \begin{cases} \beta = \omega/c & ; \; p = 0 \\ \{\beta^2 - \lambda_p^2(r_{\substack{1\\2}}, r_{\substack{3\\4}})\}^{1/2} & ; \; p \geq 1; \; \beta > \lambda_p \\ -i\{\lambda_p^2(r_{\substack{1\\2}}, r_{\substack{3\\4}}) - \beta^2\}^{1/2} & ; \; p \geq 1; \; \lambda_p > \beta \end{cases} \qquad (2.2)$$

with λ_p being the cut-off wave number corresponding to TM_{0p}-mode.

We have chosen a_i and b_i such that $a_i a_i^*$ and $b_i b_i^*$ give the incident and the reflected powers respectively in the i^{th} port. Then the scattering matrix S relates the coefficients a_i and b_i through

$$b_i = \sum_{j=1}^{3} S_{i,j} a_j; \quad i = 1,2,3 \qquad (2.3)$$

We let $H_{\phi A}$ and $H_{\phi B}$ to be the total magnetic fields in regions A and B respectively and define the scattered fields $H^s_{\phi A}$ and $H^s_{\phi B}$ as follows:

$$H^s_{\phi A}(\rho,z) = H_{\phi A}(\rho,z) - H^{+}_{\phi 1}(\rho,z) \qquad (2.4a)$$

$$H^s_{\phi B}(\rho,z) = H_{\phi B}(\rho,z) - H^{+}_{\phi 2}(\rho,z) - H^{+}_{\phi 3}(\rho,z). \qquad (2.4b)$$

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Defining the transverse electric fields $E_{\rho A}$, $E_{\rho B}$, $E^s_{\rho A}$, and $E^s_{\rho B}$ in a similar manner we can obtain $H^s_{\phi A/B}(\rho,z)$ in terms of $E^s_{\phi A/B}(\rho,0)$ as

$$H^s_{\phi \genfrac{}{}{0pt}{}{A}{B}}(\rho,z) = \mp\omega\varepsilon\int_{r_{1,2}}^{r_{3,4}} E^s_{\rho \genfrac{}{}{0pt}{}{A}{B}}(\rho',0)\rho' d\rho' \sum_{p=0}^{\infty} Z_p(r_{\genfrac{}{}{0pt}{}{1}{2}},r_{\genfrac{}{}{0pt}{}{3}{4}};\rho) \cdot Z_p(r_{\genfrac{}{}{0pt}{}{1}{2}},r_{\genfrac{}{}{0pt}{}{3}{4}};\rho')(\beta_{p,\genfrac{}{}{0pt}{}{A}{B}})^{-1}\exp(\pm i\beta_{p,\genfrac{}{}{0pt}{}{A}{B}}z) \quad (2.5)$$

We identify the coefficient b_1 in $H^s_{\phi A}$ and b_2 and b_3 in $H^s_{\phi B}$ through the relations (2.1a)-(2.1c) and, noting that the total transverse electric fields are given by the sum of the scattered and the incident fields in the corresponding regions, establish the following relation valid for arbitrary values of a_1, a_2 and a_3.

$$S_{12}a_1 + (1+S_{22})a_2 + S_{23}a_3 = n_0\{(1+S_{11})a_1 + S_{12}a_2 + S_{13}a_3\} \quad (2.6a)$$

where

$$n_0 = \{\ell n(r_3/r_1)/\ell n(r_4/r_2)\}^{1/2} \quad (2.6b)$$

In addition, if we denote $\tilde{E}_\rho(\rho,0)$ and $\approx{E}_\rho(\rho,0)$ to be the total transverse electric fields, in the plane $z = 0$, corresponding to the cases i) $a_1 = a_2 = 1$, $a_3 = 0$ and ii) $a_1 = (1-S_{33})/S_{13}$, $a_2 = 0$, $a_3 = 1$ respectively, we can establish the following <u>exact</u> relations from (2.5)

$$\{2n_0 - (1+n_0^2)S_{12}\}/S_{12} = \beta\,\ell n(r_3/r_1) \cdot\left\{\sum_{p=1}^{\infty}\left(\int_{r_2}^{r_3}\tilde{E}_\rho(\rho,0)Z_p(r_1,r_3;\rho)\rho\,d\rho\right)^2/\beta_{p,A} + \sum_{p=1}^{\infty}\left(\int_{r_2}^{r_3}\tilde{E}_\rho(\rho,0)Z_p(r_2,r_4;\rho)\rho\,d\rho\right)^2/\beta_{p,B}\right\} \Big/ \left(\int_{r_2}^{r_3}\tilde{E}_\rho(\rho,0)\,d\rho\right)^2 \quad (2.7)$$

$$\{(1-S_{33})(1-S_{11}-n_0S_{12}) - (1+n_0^2)S_{13}S_{13}\}/\{(1+S_{11})(1-S_{33})+S_{13}S_{13}\} = \beta\,\ell n(r_3/r_1)\left\{\sum_{p=1}^{\infty}\left(\int_{r_2}^{r_3}\approx{E}_\rho(\rho,0)\cdot Z_p(r_1,r_3;\rho)\rho\,d\rho\right)^2/\beta_{p,A} + \sum_{p=2}^{\infty}\left(\int_{r_2}^{r_3}\approx{E}_\rho(\rho,0)Z_p(r_2,r_4;\rho)\rho\,d\rho\right)^2/\beta_{p,B}\right\} \Big/ \left(\int_{r_2}^{r_3}\approx{E}_\rho(\rho,0)\,d\rho\right)^2 \quad (2.8)$$

The evaluation of the right hand side expressions in (2.7) and (2.8) requires a knowledge of the quantities $\tilde{E}_\rho$ and $\approx{E}_\rho$. However these expressions can be shown to be variational with respect to $\tilde{E}_\rho$ and $\approx{E}_\rho$. Because of this variational nature highly accurate results may be obtained by using the approximations

$$\tilde{E}_\rho(\rho,0) \simeq c_1/\rho \quad (2.9a)$$

$$\approx{E}_\rho(\rho,0) \simeq c_2/\rho \quad (2.9b)$$

in which case the desired expressions can be evaluated in closed form without any knowledge of the constants c_1 and c_2. Using (2.6a), (2.6b), (2.7) and (2.8) and the following two additional independent equations,

$$S_{11}S_{11}^* + S_{12}S_{12}^* + S_{13}S_{13}^* = 1 \quad (2.10)$$

$$S_{11}S_{13}^* + S_{12}S_{23}^* + S_{13}S_{33}^* = 0 \quad (2.11)$$

which follow from the symmetry and the unitary property of S, we can uniquely determine the scattering matrix which is given below:

$$S = \frac{1}{\Delta}\begin{bmatrix} -1-i(1-n_0^2-n_1^2)X_{0,1}; & -i2n_0X_{0,1}; & -i2n_1X_{0,1} \\ -i2n_0X_{0,1}; & -1+i(1-n_0^2+n_1^2)X_{0,1}; & -i2n_0n_1X_{0,1} \\ -i2n_1X_{0,1}; & -i2n_0n_1X_{0,1}; & -1+i(1+n_0^2-n_1^2)X_{0,1} \end{bmatrix} \quad (2.12a)$$

$$\Delta = 1 - i(1+n_0^2+n_1^2)X_{0,1} \quad (2.12b)$$

where n_0 is given by (2.6b) and n_1 and $X_{0,1}$ are given by the following general relationships:

$$\genfrac{}{}{0pt}{}{m_p}{n_q} = (\beta/\beta_{\genfrac{}{}{0pt}{}{p,A}{q,B}})^{1/2}\left(\ell n\frac{r_3}{r_1}\right)^{1/2}\left(\ell n\frac{r_3}{r_2}\right)^{-1} \cdot \int_{r_2}^{r_3} Z_{\genfrac{}{}{0pt}{}{p}{q}}(r_{\genfrac{}{}{0pt}{}{1}{2}},r_{\genfrac{}{}{0pt}{}{3}{4}};\rho)\,d\rho;$$

$$M \geq p \geq 1 \text{ and } N \geq q \geq 1 \quad (2.13a)$$

$$X_{M,N} = \frac{(\ell n(r_3/r_2))^2}{\beta\,\ell n(r_3/r_1)} \cdot\left\{\sum_{p=M+1}^{\infty}\left(\int_{r_2}^{r_3}Z_p(r_1,r_3;\rho)d\rho\right)^2/(i\beta_{p,A}) + \sum_{p=N+1}^{\infty}\left(\int_{r_2}^{r_3}Z_{\genfrac{}{}{0pt}{}{p}{q}}(r_2,r_4;\rho)d\rho\right)^2/(i\beta_{p,B})\right\}^{-1} \quad (2.13b)$$

The above relationships give the parameters of the equivalent circuit (Fig. 1c) when the number of propagating higher order modes in regions A and B is M and N respectively. Since the functions Z_p involve only the Bessel functions of unit order the integrals in the above expressions can be readily evaluated in closed form. We note that, when $r_3 = r_4$, our expression for the discontinuity capacitance, as given by $c_d = 1/(z_1X_{0,0}\omega)$, reduces to the known special result[1] where z_1 is the characteristic impedance of the left hand section of the coaxial line.

3. Numerical Results

Using the scattering matrix obtained in the previous section the transmission characteristics of a coaxial TEM cell of the type shown in Fig. 2a were analyzed with the aid of the equivalent network representation shown in Fig. 2b where each discontinuity is replaced by a 3-port junction and the cell is characterized by two transmission lines corresponding to TEM and TM_{01} modes. The transmission coefficient of an empty cell is plotted in Fig. 3 with the length of the cell as the parameter. The cell dimensions are $r_4 = 100$ cm, $r_3 = 80$ cm, and $r_3/r_1 = r_4/r_2 = 2.3$. We notice deep nulls in the neighborhood of points where the parameter $\beta_1 L/\pi$ is an integer, besides the nulls at the cut-off points of TM_{01} and TM_{02} modes which are at 263 MHz and 329 MHz respectively. A Small dipole type source inside the cell can be replaced by equivalent voltage sources across the transmission lines. The results for the case where the cell is excited with an internal source will be discussed at the time of the presentation.

Reference

1. R.N. Ghose, Microwave Circuit Theory and Analysis, Ch. 11, pp. 302-312, New York: McGraw-Hill Book Company, Inc., 1963.

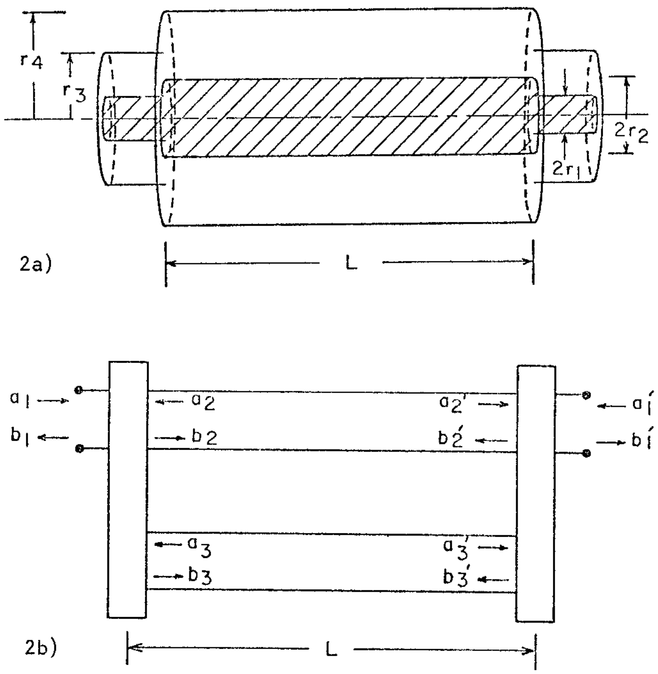
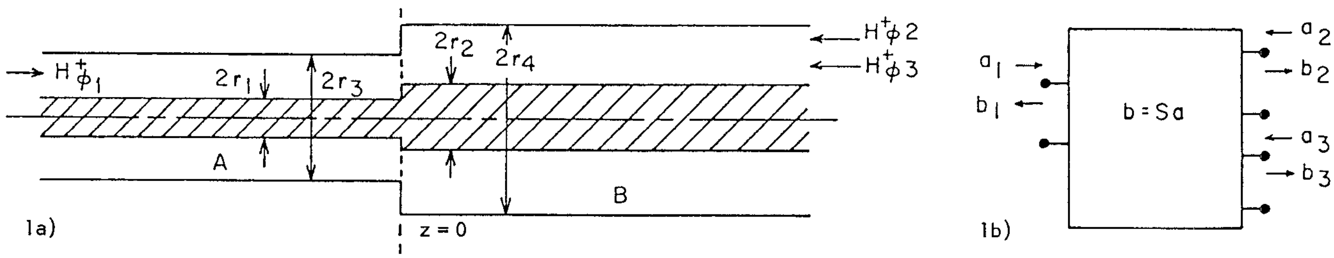
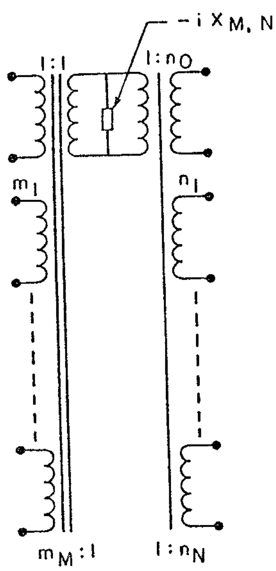


Fig. 2. a) Coaxial TEM cell.
b) Equivalent transmission line network representation of the TEM cell.





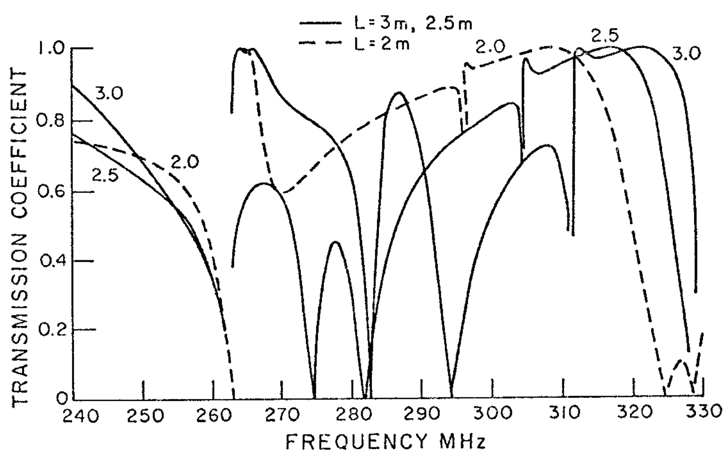


Fig. 1. a) Step discontinuity in a coaxial transmission line.
b) Scattering matrix representation of the step discontinuity.
c) Equivalent circuit of the step discontinuity.

Fig. 3. Voltage transmission coefficient vs. frequency characteristics of the coaxial TEM cell.